

Q.2 a. Prove that $\cos \alpha + \cos \left(\alpha + \frac{2\pi}{3} \right) + \cos \left(\alpha + \frac{4\pi}{3} \right) = 0$

Answer:

L.H.S.

$$\begin{aligned}
 &= \cos \alpha + \cos \left(\alpha + \frac{2\pi}{3} \right) + \cos \left(\alpha + \frac{4\pi}{3} \right) \\
 &= \cos \alpha + \cos (\alpha + 120^\circ) + \cos (\alpha + 240^\circ) \\
 &= \cos \alpha + \cos (90^\circ + \alpha + 30^\circ) + \cos (180^\circ + \alpha + 60^\circ) \\
 &= \cos \alpha - \sin (\alpha + 120^\circ) - \cos (\alpha + 60^\circ) \\
 &= \cos \alpha - (\sin \alpha \cos 30^\circ + \cos \alpha \sin 30^\circ) \\
 &= \cos \alpha - (\sin \alpha \cos 30^\circ + \cos \alpha \sin 30^\circ) - (\cos \alpha \cos 60^\circ - \sin \alpha \sin 60^\circ) \\
 &= \cos \alpha - \left(\frac{\sqrt{3}}{2} \sin \alpha + \frac{1}{2} \cos \alpha \right) - \left(\frac{1}{2} \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha \right) \\
 &= \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha + \frac{1}{2} \cos \alpha - \frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \sin \alpha \\
 &= \cos \alpha - \cos \alpha = 0
 \end{aligned}$$

b. Prove that $\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$

Answer:

L.H.S

$$\begin{aligned}
 &= \frac{\sec 8A - 1}{\sec 4A - 1} \\
 &= \frac{\frac{1}{\cos 8A} - 1}{\frac{1}{\cos 4A} - 1} \\
 &= \frac{\cos 4A(1 - \cos 8A)}{\cos 8A(1 - \cos 4A)} \\
 &= \frac{\cos 4A}{\cos 8A} \cdot \frac{2 \sin^2 4A}{2 \sin^2 2A} \\
 &= \frac{2 \sin 4A \cos 4A \cdot \sin 4A}{\cos 8A \cdot 2 \sin^2 2A} \\
 &= \frac{\tan 8A}{\tan 2A} = \text{R.H.S.}
 \end{aligned}$$

Q.3 a. Find the coefficient of x^{18} in the expansion of $\left(x^2 + \frac{3a}{x}\right)^{15}$

Answer:

Given expression is $\left(x^2 + \frac{3a}{x}\right)^{15}$

Let x^{18} occurs in the given expansion in T_{r+1}

$$T_{r+1} = {}^{15}C_r (x^2)^{15-r} \left(\frac{3a}{x}\right)^r \quad (\because T_{r+1} = n C_r x^{n-r} a^r)$$

$$\begin{aligned} &= \frac{15!}{r!(15-r)!} x^{30-2r} \frac{(3a)^r}{x^r} \\ &= \frac{15!}{r!(15-r)!} (3a)^r x^{30-2r} \quad \text{----- (i)} \end{aligned}$$

Since x^{18} occurs in T_{r+1}

$$\therefore 30 - 3r = 18$$

$$\text{or } -3r = 18 - 30$$

$$\text{or } -3r = -12$$

$$\text{or } r = 4$$

Putting $r = 4$ in equation (i), in get

$$\begin{aligned} T_5 &= \frac{15!}{4!11!} (3a)^4 \cdot x^{30-12} \\ &= \frac{15 \times 14 \times 13 \times 12 \times 11!}{4 \times 3 \times 2 \times 1 \times 11!} 81a^4 x^{18} \\ &= (1365)(81a^4) x^{18} \end{aligned}$$

\therefore Required coefficient of x^{18} in the given expansion is $110565 a^4$

- b. If the first term of an A.P is 2 and the sum of first five terms is equal to one fourth of the sum of the next five terms, then (i) show that $T_{20} = -112$
(ii) find the sum of first 30 terms.

Answer:

Let the A.P. be $a, a+d, a+2d$ -----

$$T_1 + T_2 + T_3 + T_4 + T_5 = \frac{1}{4} [T_6 + T_7 + T_8 + T_9 + T_{10}]$$

$$\therefore \frac{5}{2}(T_1 + T_5) = \frac{1}{4} \times \frac{5}{2}(T_6 + T_{10})$$

$$\begin{aligned} \text{(sum)} &= \frac{n}{2} (\text{first term} + \text{last term}) \\ \Rightarrow \frac{5}{2} [a + (a + 4d)] &= \frac{5}{8} [(a + 5d) + (a + 9d)] \end{aligned}$$

$$\Rightarrow 2a + 4d = \frac{1}{4} (2a + 14d)$$

$$\Rightarrow 4a + 8a = a + 7d$$

$$\Rightarrow d = -3a$$

$$\therefore d = -3(2) = -6 \quad (\because a = 2 \text{ is given})$$

$$\therefore a = 2 \text{ \& } d = -6$$

$$(i) \quad T_{20} = a + (20-1)d = 2 + 19(-6) = 2 - 114 = -112$$

$$\begin{aligned} (ii) \quad S_{30} &= \frac{30}{2} [(2(2) + (30-1)(-6))] \\ &= 15[4 - 174] \\ &= 15(-170) = -2550 \end{aligned}$$

Q.4 a. Show that

$$\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c)$$

Answer:

$$\text{L.H.S} = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$$

Operating $R_1 \rightarrow R_1 + R_3$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ b+c & c+a & a+b \end{vmatrix}$$

Operating $C_1 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$ we have

$$\begin{aligned}
&= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2 - a^2 & c^2 - a^2 \\ b+c & a-b & a-c \end{vmatrix} \\
&= (a+b+c)(a-b)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a^2 & -(a+b) & c+a \\ b+c & 1 & -1 \end{vmatrix} \\
&= (a+b+c)(a-b)(c-a) \begin{vmatrix} -(a+b) & c+a \\ 1 & -1 \end{vmatrix} \\
&= (a+b+c)(a-b)(c-a) |a+b \quad -c-a| \\
&= (a+b+c)(a-b)(b-c)(c-a)
\end{aligned}$$

b. Using determinants solve the following system of equations:

$$\begin{aligned}
2y - 3z &= 0 \\
x + 3y &= -4 \\
3x + 4y &= 3
\end{aligned}$$

Answer:

The given equation are

$$\begin{aligned}
0.x + 2y - 3z &= 0 \\
x + 3y + 0.z &= -4 \\
3x + 4y + 0.z &= 3
\end{aligned}$$

$$\text{Here } \Delta = \begin{vmatrix} 0 & 2 & -3 \\ 1 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix}$$

(expanding by C_3)

$$\begin{aligned}
&= -3 \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = -3(4-9) \\
&= 15 \neq 0
\end{aligned}$$

Because \neq therefore, system has unique solution

$$\begin{aligned}
\Delta_1 &= \begin{vmatrix} 0 & 2 & -3 \\ -4 & 3 & 0 \\ 3 & 4 & 0 \end{vmatrix} \\
&= 0 - 2(0-0) - 3(-16-19)
\end{aligned}$$

$$= 75$$

$$\Delta_2 = \begin{vmatrix} 0 & 0 & -3 \\ 1 & -4 & 0 \\ 3 & 3 & 0 \end{vmatrix}$$

$$= -3(3+12) = -45$$

$$\Delta_3 = \begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & -4 \\ 3 & 4 & 3 \end{vmatrix}$$

$$= -2(3+12) = -30$$

$$\therefore x = \frac{\Delta_1}{\Delta} = \frac{75}{15} = 5$$

$$y = \frac{\Delta_2}{\Delta} = \frac{-45}{15} = -3$$

$$z = \frac{\Delta_3}{\Delta} = \frac{-30}{15} = -2$$

$$\therefore x = 5, y = -3, z = -2$$

- Q.5** a. Find the equation of the right bisector of the segment joining A(1, 1) and B(2, 3)

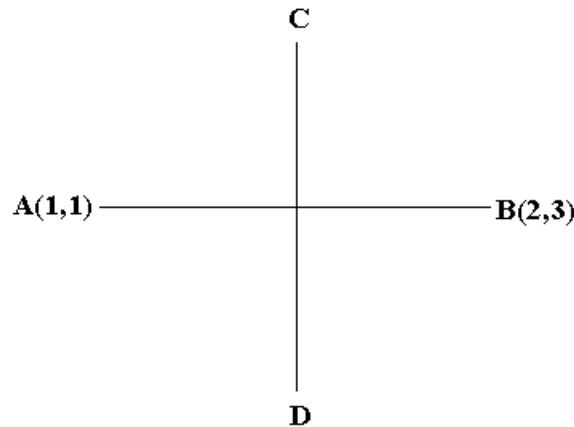
Answer:

Let AB be a segment joining the points A(1,1) ,B(2,3) & CD be its right bisector. Then CD passes through the midpoint of AB is

$$\left(\frac{1+2}{2}, \frac{1+3}{2} \right)$$

$$\left(\frac{3}{2}, 2 \right)$$

This CD passes through $\left(\frac{3}{2}, 2 \right)$



Also slope of AB = $\frac{3-1}{2-1} = 2$

Now CD \perp AB

\therefore equation of CD which passes through $(\frac{3}{2}, 2)$

and has slope $-1/2$ is

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{1}{2}\left(x - \frac{3}{2}\right)$$

$$4y - 8 = -2x + 3$$

$$2x + 4y - 11 = 0$$

Which is the required equation.

- b. Find the equation of the lines through the origin and making an angle of 60° with the line $x + y\sqrt{3} + 3\sqrt{3} = 0$

Answer:

Let m be the slope of any one of the required lines through $(0,0)$

Then the equation is $y - 0 = m(x - 0)$

$$y = mx \quad \text{_____ (i)}$$

also slope of $x + \sqrt{3}y + 3\sqrt{3} = 0$ is $-\frac{1}{\sqrt{3}}$

Angle between the two line is given to be 60°

$$\tan 60^\circ = \left| \frac{m + \frac{1}{\sqrt{3}}}{1 + m\left(-\frac{1}{\sqrt{3}}\right)} \right| \quad \left(\text{Using } \tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \right)$$

$$\Rightarrow \pm\sqrt{3} = \frac{\sqrt{3}m+1}{\sqrt{3}-m}$$

Taking + ve sign, we here $3 - \sqrt{3}m = \sqrt{3}m + 1$

$$-2\sqrt{3}m = -2$$

$$m = \frac{1}{\sqrt{3}}$$

$$\therefore \text{from (i), } y = \frac{1}{\sqrt{3}}x$$

Taking -ve sign, we here

$$-3 + \sqrt{3}m = \sqrt{3}m + 1$$

Here m does not here a finite value, then the line is $x = 0$.

(\therefore The line passes through the origin).

- Q.6** a. Find the equation of the circle which passes through the points (5, -8), (2, -9) and (2, 1). Find also the co-ordinates of its centre and radius.

Answer:

Let the required equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{_____ (i)}$$

(i) Passes through (5, -8)

$$(5)^2 + (-8)^2 + 2g(5) + 2f(-8) + c = 0$$

$$25 + 64 + 10g - 16f + c = 0$$

$$10g - 16f + c = -89 \quad \text{_____ (ii)}$$

(i) Passes through (2, -9)

$$(2)^2 + (-9)^2 + 2g(2) + 2f(-9) + c = 0$$

$$4g - 18f + c = -85 \quad \text{_____ (iii)}$$

(i) Passes through (2, 1)

$$(2)^2 + (1)^2 + 2g(2) + 2f(1) + c = 0$$

$$4g + 2f + c = -5 \quad \text{_____ (iv)}$$

Solving (ii) & (iii)

$$10g - 16f + c = -89$$

$$4g - 18f + c = -85$$

$$6g + 2f = -4 \quad \text{_____ (v)}$$

Solving (iii) & (iv)

$$4g - 18f + c = -85$$

$$4g + 2f + c = -5$$

$$-20f = -80$$

$$f = 4$$

Substituting the value of 'f' in (v)

$$6g + 2(4) = -4$$

$$6g = -4 - 8 = -12$$

$$g = -2$$

Substituting the value of 'g' and 'f' in equation (iv)

$$4(-2) + 2(4) + c = 14$$

$$c = -5$$

substituting the value of g, f, c in equation (i)

$$4(-2) + 2(4) + c = -5$$

$$c = -5$$

Substituting the values of g, f, c in equation (i)

$$x^2 + y^2 - 4x + 8y - 5 = 0$$

$$c(-g, -f) = c(2, -4) \quad \& \quad r = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{4 + 16 + 5} = \sqrt{25} = 5$$

- b. Find the length of major and minor axis, eccentricity, the co-ordinates of vertices and foci, directrices and the length latus rectum of the ellipse $3x^2 + 2y^2 = 6$.

Answer:

The given ellipse is $3x^2 + 2y^2 = 6$

Comparing it with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$a^2 = 2, b^2 = 3, a^2 < b^2 \text{ u } a < b$$

$$\text{Length of the major axis} = 2b = 2\sqrt{3}$$

$$\text{Length of the minor axis} = 2a = 2\sqrt{2}$$

$$a^2 = b^2(1 - e^2)$$

$$2 = 3(1 - e^2)$$

$$\frac{2}{3} = 1 - e^2$$

$$e^2 = 1 - 1 - \frac{2}{3} = \frac{1}{3}$$

$$e = \frac{1}{\sqrt{3}}$$

co-ordinates of vertices are $(0, \pm b)$ ie, $(0, \pm \sqrt{3})$

co-ordinates of foci are $(0, \pm be)$ ie $(0, \pm \frac{1}{\sqrt{3}}\sqrt{3})$

$$= (0, \pm 1)$$

Equation of the directrices are $y = \pm \frac{b}{e}$

$$y = \pm \frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} \pm 3$$

$$\text{length of latus x here} = \frac{2b^2}{a} = \frac{2(2)}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

Q.7 a. If $y = \log(x + \sqrt{1+x^2})$, Prove that $(1+x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$

Answer:

$$\sin y = x \sin(a+y)$$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating both sides w.r.t 'y'

$$\frac{dx}{dy} = \frac{\sin(a+y) \frac{d}{dy} \sin y - \sin y \frac{d}{dy} \sin(a+y)}{\sin^2(a+y)}$$

$$\frac{\sin(a+y) \cos y - \cos(a+y) \sin y}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\sin(a+y-y)}{\sin^2(a+y)} = \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{\sin a}{\sin^2(a+y)}} = \frac{\sin^2(a+y)}{\sin a}$$

b. Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}$ at the point where it cuts x-axis.

Answer:

$$\text{The given curve is } y = \frac{x-7}{(x-2)(x-3)} \text{ (i)}$$

This curve cut x-axis; $\therefore y = 0$

$$\text{We get } 0 = \frac{x-7}{(x-2)(x-3)}$$

$$\therefore x = 7$$

This (i) cuts at x – axis at (7, 0) Diff. (i) w.r.t 'x' both sides

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6) \frac{d}{dx}(x - 7) - (x - 7) \frac{d}{dx}(x^2 - 5x + 6)}{(x^2 - 5x + 6)^2}$$

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6)(1) - (x - 7)(2x - 5)}{(x^2 - 5x + 6)^2}$$

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6) - (2x^2 - 5x - 14x + 35)}{(x^2 - 5x + 6)^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 5x + 6 - 2x^2 + 19x - 35}{(x^2 - 5x + 6)^2}$$

$$\frac{dy}{dx} = \frac{-x^2 + 14x - 29}{(x^2 - 5x + 6)^2}$$

$$\left(\frac{dy}{dx}\right)_{(7,0)} = \frac{-(7)^2 + 14(7) - 29}{((7)^2 - 5(7) + 6)^2} = \frac{-49 + 98 - 29}{(49 - 35 + 6)^2}$$

$$\frac{-78 + 98}{(20)^2} = \frac{20}{(20)^2} = \frac{1}{20}$$

New equation of tangent at the point (7, 0) sharing slop $\frac{1}{20}$ is

$$y - 0 = \frac{1}{20}(x - 7) \quad (\text{Using } y - y_1 = m(x - x_1))$$

$$20y = x - 7$$

$$x - 20y - 7 = 0$$

Q.8 a. Evaluate $\int \frac{4x + 1}{x^2 + 3x + 2} dx$

Answer:

$$\text{Let } I = \int \frac{4x + 1}{x^2 + 3x + 2} dx$$

$$4x + 1 = \lambda \frac{d}{dx}(x^2 + 3x + 2) + \mu$$

$$4x + 1 = 2\lambda x + 3\lambda + \mu$$

Comparing the coefficients.

$$4 = 2\lambda, \quad 1 = 3\lambda + \mu$$

$$\lambda = 2, \quad 1 = 3(2) + \mu$$

$$\mu = 1 - 6 = -5$$

Substituting the values of 'λ' & 'μ' in equation (i)

$$4x + 1 = 2(2x + 3) - 5$$

$$\begin{aligned} I &= \int \frac{2(2x+3)-5}{x^2+3x+2} dx \\ &= 2 \int \frac{2x+3}{x^2+3x+2} dx - 5 \int \frac{1}{x^2+3x+2} dx \\ &= 2 \int \frac{2x+3}{x^2+3x+2} dx - 5 \int \frac{1}{x^2+3x+\frac{9}{4}-\frac{9}{4}+2} dx \\ &= 2 \int \frac{2x+3}{x^2+3x+2} dx - 5 \int \frac{1}{\left(x^2+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} dx \\ &= 2 \log|x^3+3x+2| - 5 \frac{1}{2\left(\frac{1}{2}\right)^2} \log \left| \frac{x+\frac{3}{2}-\frac{1}{2}}{x+\frac{3}{2}+\frac{1}{2}} \right| + c \\ &= 2 \log|x^3+3x+2| - 5 \log \left| \frac{2x+2}{2x+4} \right| + c \\ &= 2 \log|x^3+3x+2| - 5 \log \left| \frac{x+2}{x+4} \right| + c \end{aligned}$$

b. Evaluate $\int_0^{\frac{\pi}{4}} \log(1 + \tan x) dx$

Answer:

$$\text{Let } I = \int_0^{\pi/4} \log(1 + \tan x) dx \quad \text{_____ (i)}$$

$$\text{(Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^{\pi/4} \log(1 + \tan(\frac{\pi}{4} - x)) dx$$

$$= \int_0^{\pi/4} \log\left(1 + \frac{1 - \tan x}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left(\frac{1 + \tan x + 1 - \tan x}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1 + \tan x}\right) dx$$

$$= \int_0^{\pi/4} (\log 2 - \log(1 + \tan x)) dx \quad \text{_____ (ii)}$$

$$= \text{Adding (i) \& (ii)}$$

$$2I = \int_0^{\pi/4} \log 2 dx = \log 2 \int_0^{\pi/4} 1 dx$$

$$= \frac{\pi}{8} \log 2$$

Q.9 a. Solve the initial value problem

$$x \frac{dy}{dx} + \cot y = 0, \text{ when } y(\sqrt{2}) = \pi/4$$

Answer:

$$x \frac{dy}{dx} + \cot y = 0$$

$$\frac{1}{\cot y} dy + \frac{1}{x} dx = 0$$

(Using variable separated)

Integrating both sides

$$\int \tan dy + \int \frac{1}{x} dx = \int 0 dx$$

$$\log|\sec y| + \log|x| = \log|c|$$

Taking antilog both sides

$$X \sec y = c \quad \text{_____ (i)}$$

$$\text{Also } y(\sqrt{2}) = \pi/4$$

u. when $x = \sqrt{2}$, $y = \frac{\pi}{4}$

put in (i)

$$\sqrt{2} \sec \frac{\pi}{4} = c$$

$$\sqrt{2} \cdot \sqrt{2} = c$$

Put this value in (i)

$$x \sec y = 2$$

b. Solve $(x^2 + xy)dy = (x^2 + y^2)dx$

Answer:

$$(x^2 + xy)dy = (x^2 + y^2)dx$$

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy} \text{ (i)}$$

Clearly it is a homogeneous equation

$$\therefore \text{ Put } y = vx \text{ (ii)}$$

Differentiating both sides w.r.t. (x)

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \text{ (iii)}$$

Put the values of (ii) and (iii) in (i) we get

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2x^2}{x^2 + x(vx)} = \frac{x^2 + v^2x^2}{x^2 + vx^2}$$

$$v + x \frac{dv}{dx} = \frac{1 + v^2}{1 + v}$$

$$x \frac{dv}{dx} = \frac{1 + v^2}{1 + v} - v = \frac{1 + v^2 - v - v^2}{1 + v} = \frac{1 - v}{1 + v}$$

$$\frac{1 - v}{1 + v} dv = \frac{1}{x} dx$$

Integrating both sides

$$\int \frac{1 + v}{1 - v} dv = \int \frac{1}{x} dx$$

$$\int -1 + \frac{2}{1 - v} dv = \int \frac{1}{x} dx$$

$$-v + 2 \frac{\log|1 - v|}{-1} = \log|x| + c$$

$$-\frac{y}{x} - \log\left|1 - \frac{y}{x}\right| - \log|x| = c$$

$$-\frac{y}{x} - \log\left|\left(\frac{x - y}{x}\right)^2 \cdot x\right| = c$$

$$-\frac{y}{x} - \log \left| \frac{(x-y)^2}{x} \right| = c$$
$$\frac{y}{x} + \log \left| \frac{(x-y)^2}{x} \right| = -c$$

Text Books

- I. Applied Mathematics for Polytechnics, H. K. Dass, 8th Edition, CBS Publishers & Distributors.
- II. A Text book of Comprehensive Mathematics Class XI, Parmanand Gupta, Laxmi Publications (P) Ltd, New Delhi.
- III. Engineering Mathematics, H. K. Dass, S, Chand and Company Ltd, 13th Edition, New Delhi.